Year 13 Mock Exam Answers

1) a) $f(x) = \dfrac{2x^2+4}{(x-2)^2(x+4)} = \dfrac{A}{(x-2)} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x+4)}$

$$= \frac{A(x-2)(x+4) + B(x+4) + C(x-2)^2}{(x-2)^2(x+4)}$$

$$\therefore \; 2x^2+4 = A(x-2)(x+4) + B(x+4) + C(x-2)^2$$

Let $\underline{x=2}$

$8+4 = 0A + 6B + 0C$

$12 = 6B$

$2 = B$

Let $\underline{x=-4}$

$32+4 = 36C$

$36 = 36C$

$1 = C$

Let $\underline{x=0}$

$4 = -8A + 4B + 4C$

$4 = -8A + 8 + 4$

$8A = 8+4-4$

$8A = 8$

$A = 1$

$$\therefore \; f(x) = \frac{1}{(x-2)} + \frac{2}{(x-2)^2} + \frac{1}{(x+4)}$$

b) $f'(x) = -(x-2)^{-2} - 4(x-2)^{-3} - (x+4)^{-2}$ $\quad = (x-2)^{-1} + 2(x-2)^{-2} + (x+4)^{-1}$

$$= -\frac{1}{(x-2)^2} - \frac{4}{(x-2)^3} - \frac{1}{(x+4)^2}$$

$$f'(0) = -\frac{1}{4} - \frac{4}{(-8)} - \frac{1}{16}$$

$$= -\frac{1}{4} + \frac{1}{2} - \frac{1}{16}$$

$$= \frac{3}{16}$$

2) $2x^3 + 6xy^2 - y^4 = 27$

$$6x^2 + 12xy\frac{dy}{dx} + 6y^2 - 4y^3\frac{dy}{dx} = 0$$

$$12xy\frac{dy}{dx} - 4y^3\frac{dy}{dx} = -6x^2 - 6y^2$$

$$4y\frac{dy}{dx}\left(3x - y^2\right) = -6\left(x^2 + y^2\right)$$

$$\frac{dy}{dx} = \frac{-6(x^2+y^2)}{4y(3x-y^2)}$$

$$\frac{dy}{dx} = \frac{-3(x^2+y^2)}{2y(3x-y^2)}$$

$\underline{6xy^2}$

$u = 6x \quad v = y^2$

$\frac{du}{dx} = 6 \quad \frac{dv}{dx} = 2y\frac{dy}{dx}$

$\Rightarrow 6x2y\frac{dy}{dx} + 6y^2$

$= 12xy\frac{dy}{dx} + 6y^2$

At $(2,1)$

$$\frac{dy}{dx} = \frac{-3(4+1)}{2(6-1)} = \frac{-15}{10} = \frac{-3}{2}$$

$\therefore$ Gradient of normal $= +\frac{2}{3}$ because

$\left(-\frac{3}{2}\right) \times \frac{2}{3} = -1$

$\therefore$ Eqn of normal at $(2,1)$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{2}{3}(x - 2)$$

$$3y - 3 = 2x - 4$$

$$3y = 2x - 1$$

3) $2 + 3\cos 2\theta = \cos\theta$ $\qquad\qquad \cos 2\theta = 2\cos^2\theta - 1$

$$2 + 3(2\cos^2\theta - 1) = \cos\theta$$

$$2 + 6\cos^2\theta - 3 = \cos\theta$$

$$6\cos^2\theta - \cos\theta - 1 = 0$$

$$(3\cos\theta + 1)(2\cos\theta - 1) = 0$$

either $\qquad\qquad$ or

$3\cos\theta + 1 = 0$ $\qquad\qquad$ $2\cos\theta - 1 = 0$

$\cos\theta = \frac{-1}{3}$ $\qquad\qquad$ $\cos\theta = \frac{1}{2}$

$\alpha = 70{\cdot}5^\circ$ $\qquad\qquad$ $\alpha = 60^\circ$

Cos -ve 2nd and 3rd $\qquad$ Cos +ve 1st and 4th

$\theta = 109{\cdot}5^\circ, 250{\cdot}5^\circ$ $\qquad\qquad$ $\theta = 60^\circ, 300^\circ$

$$\therefore\ \theta = 60^\circ, 109{\cdot}5^\circ, 250{\cdot}5^\circ, 300^\circ$$

4) a) $4\sin x + 3\cos x = \sqrt{4^2+3^2}\left[\frac{4}{5}\sin x + \frac{3}{5}\cos x\right]$

$= 5\sin(x+\alpha)$

where $\cos\alpha = \frac{4}{5}$

$\alpha = 36.9^\circ$

$\therefore \quad 4\sin x + 3\cos x = 5\sin(x + 36.9^\circ)$

b) $\dfrac{1}{4\sin x + 3\cos x + 7}$

$= \dfrac{1}{5\sin(x+36.9^\circ) + 7}$

Greatest value occurs when $\sin(x+36.9^\circ) = -1$

$= \dfrac{1}{5(-1)+7}$

$= \dfrac{1}{2}$

5) Area $= \int_0^{\pi/2} \sin x \, dx$

$= \left[-\cos x\right]_0^{\pi/2}$

$= \left(-\cos \frac{\pi}{2}\right) - \left(-\cos 0\right)$

$= -0 + 1$

$= 1 \text{ units}^2$

6) Curves intersect when

$x^2 + 4 = 12 - x^2$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

$\therefore x = 2$

$y = 2^2 + 4$

$y = 8$

$(2, 8)$

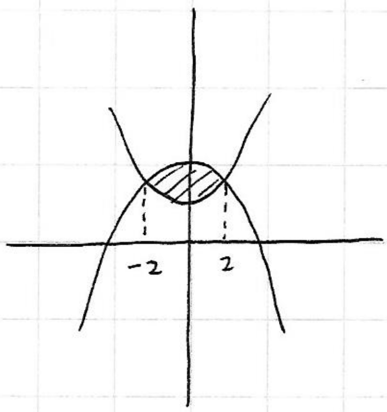
$x = -2$

$y = (-2)^2 + 4$

$y = 4 + 4$

$y = 8$

$(-2, 8)$



Area shaded $= \int_{-2}^{2} 12 - x^2 \, dx - \int_{-2}^{2} x^2 + 4 \, dx$

$= \int_{-2}^{2} 12 - 4 - x^2 - x^2 \, dx$

$= \int_{-2}^{2} 8 - 2x^2 \, dx$

$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$

$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$

$$= 16 - \frac{16}{3} + 16 - \frac{16}{3}$$

$$= \cancel{32 \text{ units}^2}$$

$$= 32 - \frac{32}{3}$$

$$= \frac{96}{3} - \frac{32}{3}$$

$$= \frac{64}{3} \text{ units}^2$$

7) $y = ax^4 + bx^3 + 18x^2$

a) $\frac{dy}{dx} = 4ax^3 + 3bx^2 + 36x$

$\frac{d^2y}{dx^2} = 12ax^2 + 6bx + 36 = 0$ at point inflection

Now $(1, 11)$ is inflection pt ... $x = 1$

$\therefore \quad 12a(1^2) + 6b(1) + 36 = 0$

$12a + 6b + 36 = 0$

$2a + b + 6 = 0 \quad$ —— 1

b) Now for the curve itself

$(1, 11)$ must lie on curve ... so ...

$11 = a(1^4) + b(1^3) + 18(1^2)$

$11 = a + b + 18$

$-b - 7 = a \quad (*)$

Sub $(*)$ into 1

$2(-b-7) + b + 6 = 0$

$-2b - 14 + b + 6 = 0$

$-8 = b$

$(*) \Rightarrow \quad -(-8) - 7 = a$

$8 - 7 = a$

$1 = a$

$\therefore \quad \dfrac{d^2y}{dx^2} = 12x^2 + 6(-8)x + 36$

$12x^2 - 48x + 36 = 0$ at inflection points

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3$ or $x = 1 \rightarrow (1, 11)$ GIVEN!

$y = 3^4 - 8(3^3) + 18(3^2)$

$y = 81 - 216 + 162$

$y = 27 \quad \therefore (3, 27)$ is point of inflection

c) $\dfrac{dy}{dx} = 4ax^3 + 3bx^2 + 36x$

$a = 1 \quad b = -8$

$\dfrac{dy}{dx} = 4x^3 - 24x^2 + 36x = 0$ at SP's

$x^3 - 6x^2 + 9x = 0$

$x(x^2 - 6x + 9) = 0$

$x = 0$ or $x^2 - 6x + 9 = 0$

$y = 1x^4 - 8x^3 + 18x^2$

$y = 0 - 0 + 0$

$y = 0$

$\therefore (0,0)$

$(x-3)(x-3) = 0$

$x = 3$

$y = 3^4 - 8(3^3) + 18(3^2)$

$y = 81 - 216 + 162$

$y = 27$

$(3, 27)$

point of inflection as before

Must be SADDLE POINT

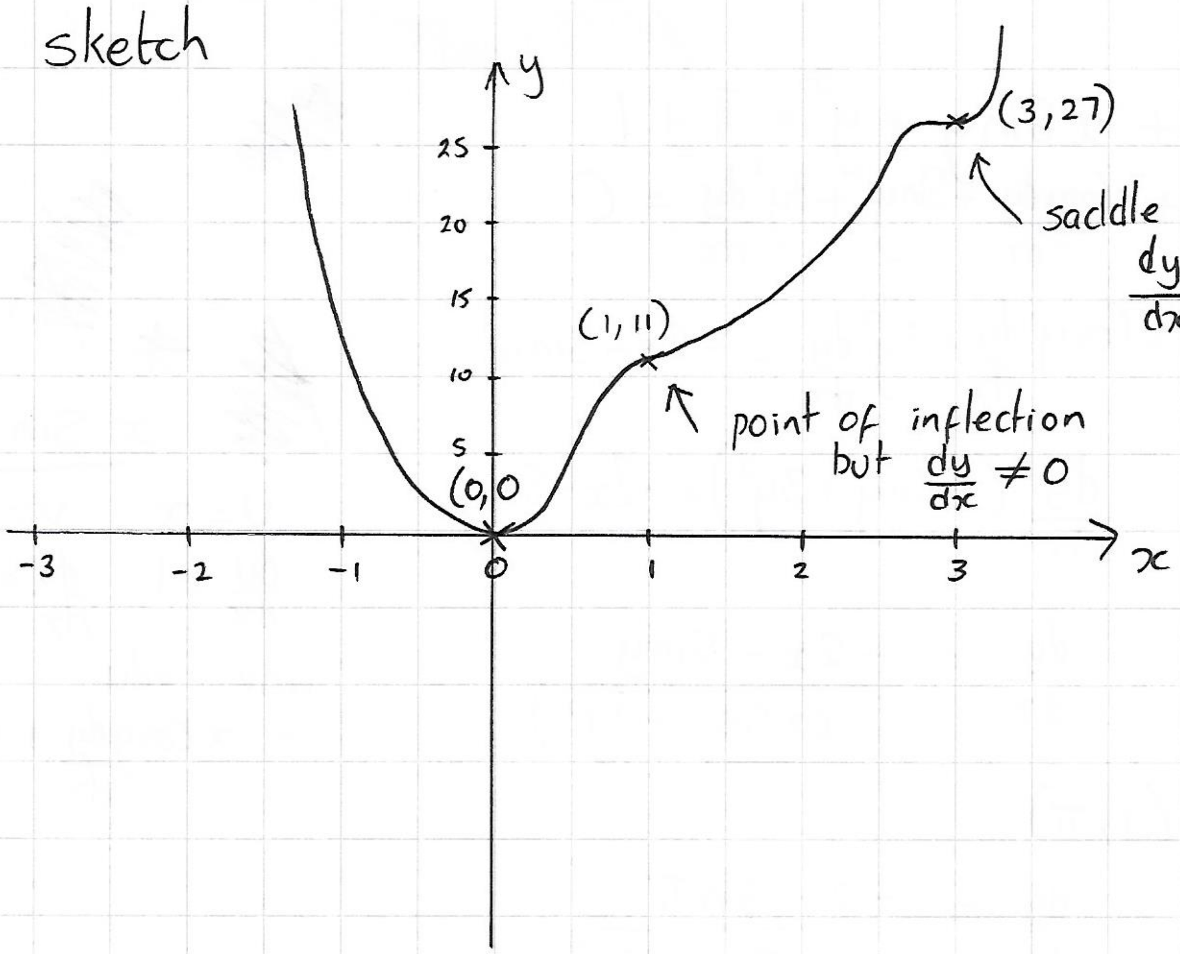
$\dfrac{d^2y}{dx^2} = 12x^2 - 48x + 36$

$= 0 - 0 + 36$

$= +36$

$\therefore (0,0)$ is LOCAL MIN

sketch


y
25
20
15
10
5
(3,27)
saddle point
$\frac{dy}{dx} = 0$
(1,11)
point of inflection
but $\frac{dy}{dx} \neq 0$
(0,0)
-3
-2
-1
0
1
2
3
x

8) $x^2 + x\sin y + y^3 = \pi^3 + 1$

$2x + x\cos y\frac{dy}{dx} + \sin y + 3y^2\frac{dy}{dx} = 0$

$x\cos y\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -2x - \sin y$

$\frac{dy}{dx}\left(x\cos y + 3y^2\right) = -2x - \sin y$

$\frac{dy}{dx} = \frac{-2x - \sin y}{(x\cos y + 3y^2)}$

$x\sin y$

$u = x \quad v = \sin y$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \cos y\frac{dy}{dx}$

$udv + vdu$

$= x\cos y\frac{dy}{dx} + (1)\sin y$

At $(1, \pi)$

$\frac{dy}{dx} = \frac{-2 - \sin\pi}{(1\cos\pi + 3\pi^2)}$

$= \frac{-2 - 0}{-1 + 3\pi^2}$

$\frac{dy}{dx} = \frac{2}{1 - 3\pi^2} = -0.0699$

9) a) $I = \int x e^{-x} \, dx$

$u = x \qquad \frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = 1 \qquad v = \frac{e^{-x}}{-1}$

$du = dx$

$$
\begin{aligned}
I &= uv - \int v\,du \\
&= -xe^{-x} - \int -e^{-x}\,dx \\
&= -xe^{-x} + \int e^{-x}\,dx \\
&= -xe^{-x} - e^{-x} + C
\end{aligned}
$$

b) $I = \int_{\frac{1}{2}}^{1} x(2x-1)^9 \, dx$

$u = 2x - 1 \qquad u + 1 = 2x$

$\frac{du}{dx} = 2 \qquad \frac{u+1}{2} = x$

$\frac{du}{2} = dx$

Limits

$x = 1 \qquad x = \frac{1}{2}$

$u = 2 - 1 \qquad u = 1 - 1$

$\boxed{u = 1} \qquad \boxed{u = 0}$

$$
\begin{aligned}
\therefore \quad I &= \int_0^1 \frac{(u+1)}{2} u^9 \frac{du}{2} \\
&= \frac{1}{4} \int_0^1 u^{10} + u^9 \, du \\
&= \frac{1}{4} \left[\frac{u^{11}}{11} + \frac{u^{10}}{10} \right]_0^1 \\
&= \frac{1}{4} \left[\left(\frac{1}{11} + \frac{1}{10} \right) - (0+0) \right] \\
&= \frac{1}{4} \left[\frac{10}{110} + \frac{11}{110} \right] \\
&= \frac{1}{4} \left[\frac{21}{110} \right] \\
&= \frac{21}{440}
\end{aligned}
$$

10) a) $f(x) = \sin^{-1}x - 2x^{3/2} + 1$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - 3x^{1/2} + 0 \qquad = 0 \text{ at SP's}$$

This is a standard result to learn

$$\frac{1}{\sqrt{1-x^2}} - 3\sqrt{x} = 0$$

$\times\sqrt{1-x^2}$ $\quad 1 - 3\sqrt{x}\sqrt{1-x^2} = 0$

$$1 = 3\sqrt{x}\sqrt{1-x^2}$$

square $\quad 1 = 9x(1-x^2)$

$$1 = 9x - 9x^3$$

$$9x^3 - 9x + 1 = 0 \qquad \text{QED}$$

11) $x = \frac{1}{t}$ $\qquad y = t^2$

$x = t^{-1}$

a) $\frac{dx}{dt} = -t^{-2}$ $\qquad \frac{dy}{dt} = 2t$

$= -\frac{1}{t^2}$

$\frac{dt}{dx} = -t^2$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2t \times (-t^2)$$

$$= -2t^3$$

$M = -2p^3$ at P

Now at $P(\frac{1}{p}, p^2)$ $\qquad$ Equation of tangent is

$$y - y_1 = M(x - x_1)$$

$$y - p^2 = -2p^3(x - \frac{1}{p})$$

$$y - p^2 = -2p^3x + \frac{2p^3}{p}$$

$$y - p^2 = -2p^3x + 2p^2$$

$$y + 2p^3x - 3p^2 = 0$$

QED.

b) Tangent crosses x axis, $y = 0$

$$0 + 2p^3x - 3p^2 = 0$$

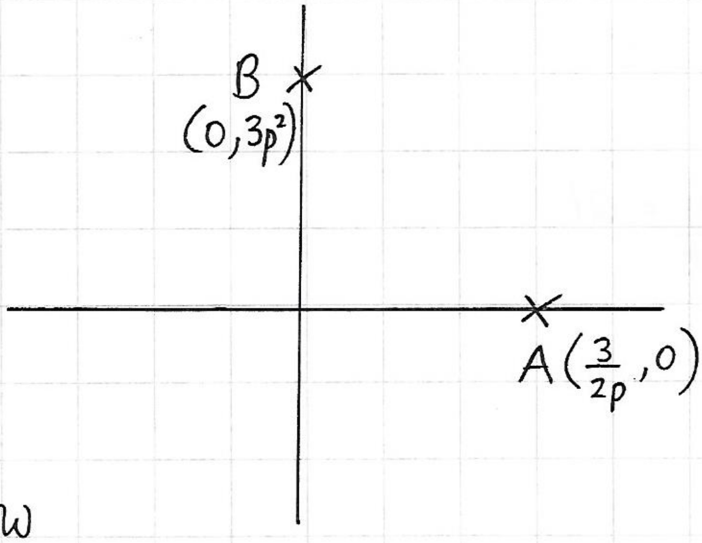
$\div p^2$ $\qquad 2px - 3 = 0$

$$x = \frac{3}{2p}$$

Tangent crosses y axis, $x = 0$

$$y - 3p^2 = 0$$

$$y = 3p^2$$



Now

PB

$P\left(\frac{1}{p}, p^2\right)$ $\quad$ $B\left(0, 3p^2\right)$

$x_1 \; y_1 \quad\quad x_2 \; y_2$

$$PB = \sqrt{(3p^2 - p^2)^2 + \left(0 - \frac{1}{p}\right)^2}$$

$$= \sqrt{4p^4 + \frac{1}{p^2}}$$

$$PB^2 = 4p^4 + \frac{1}{p^2}$$

PA

$P\left(\frac{1}{p}, p^2\right)$ $\quad$ $A\left(\frac{3}{2p}, 0\right)$

$x_1 \; y_1 \quad\quad x_3 \; y_3$

$$PA = \sqrt{(0 - p^2)^2 + \left(\frac{3}{2p} - \frac{1}{p}\right)^2}$$

$$= \sqrt{p^4 + \left(\frac{3}{2p} - \frac{2}{2p}\right)^2}$$

$$= \sqrt{p^4 + \frac{1}{4p^2}}$$

$$PA^2 = p^4 + \frac{1}{4p^2}$$

$\underline{\times 4}$

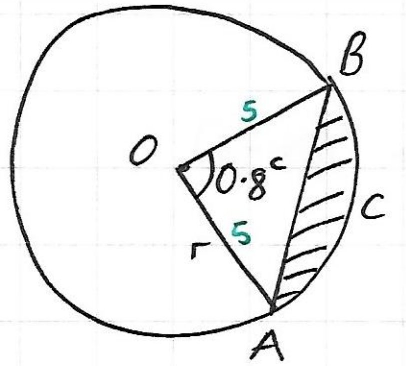
$$4PA^2 = 4p^4 + \frac{1}{p^2}$$

$$\therefore \; PB^2 = 4PA^2$$

square root $\quad PB = 2PA$

QED.

12)



Arc ACB = 4cm

a) Arc ACB

$$S = r\theta$$

$$4 = r(0.8)$$

$$\frac{4}{0.8} = r$$

$$5\text{ cm} = r$$

b) Area $/\!/\!/$ = area sector OAB − area $\triangle$ OAB

$$= \frac{1}{2}r^2\theta - \frac{1}{2}ab\sin C$$

$$= \frac{1}{2}\times 5^2 \times 0.8 - \frac{1}{2}\times 5 \times 5 \sin 0.8^c$$

$$= 10 - 8.97$$

$$= 1.03\text{ cm}^2$$

13) a) (i) $y = (5x^2 - 3x)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(5x^2 - 3x)^{-1/2} \times (10x - 3)$$

$$= \frac{(10x-3)}{2\sqrt{(5x^2-3x)}}$$

(ii) $y = \sin^{-1}(7x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(7x)^2}} \times 7$$

$$= \frac{7}{\sqrt{1-49x^2}}$$

(iii) $y = e^{3x} \ln x$

$u = e^{3x}$ $\quad v = \ln x$

$\frac{du}{dx} = 3e^{3x}$ $\quad \frac{dv}{dx} = \frac{1}{x}$

$u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{x}e^{3x} + 3e^{3x}\ln x$$

$$= e^{3x}\left(\frac{1}{x} + 3\ln x\right)$$

$$= e^{3x}\left(\frac{1}{x} + \frac{3x\ln x}{x}\right)$$

$$= e^{3x}\left(\frac{1+3x\ln x}{x}\right)$$

$$= \frac{e^{3x}(1+3x\ln x)}{x}$$

NOT really needed

b) $\cot x = \dfrac{\cos x}{\sin x}$

Let $y = \dfrac{\cos x}{\sin x}$

$u = \cos x \qquad v = \sin x$

$\dfrac{du}{dx} = -\sin x \qquad \dfrac{dv}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{1}{v^2}\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]$$

$$= \frac{1}{\sin^2 x}\left[\sin x(-\sin x) - \cos x(\cos x)\right]$$

$$= \frac{1}{\sin^2 x}\left[-\sin^2 x - \cos^2 x\right]$$

$$= \frac{-1}{\sin^2 x}\left[\underbrace{\cancel{\sin^2 x + \cos^2 x}}_{1}\right]$$

$$= -\frac{1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

QED

14) a) (i) $\int \cos\left(\frac{4x+5}{3}\right) dx$

$$= \frac{\sin\left(\frac{4x+5}{3}\right)}{4/3} + C$$

$$= \frac{3}{4}\sin\left(\frac{4x+5}{3}\right) + C$$

(ii) $\int e^{2x+9}\, dx$

$$= \frac{e^{2x+9}}{2} + C$$

(iii) $\int \frac{3}{(7-2x)^6}\, dx$

$$= 3\int (7-2x)^{-6}\, dx$$

$$= \frac{3(7-2x)^{-5}}{(-5)\times(-2)} + C$$

$$= \frac{3}{10(7-2x)^5} + C$$