Stationary and Inflexion Points : Answers

1)
$$y = ax^2 + 12x + 1$$
$$\frac{dy}{dx} = 2ax + 12 = 0 \quad \text{at} \quad SP's$$
$$2ax = -12$$
$$\cancel{x} = \frac{-12}{2x}$$

when $x = 2$
$$a = \frac{-12}{4}$$
$$a = -3$$

$\therefore$ Curve is
$$y = -3x^2 + 12x + 1$$

To identify the nature
$$\frac{dy}{dx} = -6x + 12$$
$$\frac{d^2y}{dx^2} = -6 \quad \therefore \text{ the point is a local max.}$$

2)
$$y = ax^2 + bx + c$$

There are 3 unknowns a, b and c.
We need 3 equations !!! If need be, solve simultaneously

First Max at $(2,18)$

$$\frac{dy}{dx} = 2ax + b = 0 \quad \text{at} \quad SP's.$$

$x = 2$
$$4a + b = 0 \quad — \quad 1$$

Second

If there is a max at (2,18), then (2,18) lies on the original curve!

$$y = ax^2 + bx + c$$

$x = 2$
$y = 18$

$$18 = a(2^2) + b(2) + c$$
$$18 = 4a + 2b + c \quad — \quad (2)$$

Third

The point (0,10) is also on the curve.

$$y = ax^2 + bx + c$$

$x = 0$
$y = 10$

$$10 = a(0^2) + b(0) + c$$
$$10 = c \quad — \quad (3)$$

No need for simultaneaus work yet.

Sub (3) into (2)

$$(2) \Rightarrow \quad 18 = 4a + 2b + 10$$
$$8 = 4a + 2b$$
$$4 = 2a + b \quad — \quad (4)$$

Now solve (4) and (1) simultaneously

$$4 = 2a + b \quad — (4)$$
$$0 = 4a + b \quad — (1)$$

from (4) $\quad 4 - 2a = b$

$\therefore$ (1) $\Rightarrow \quad 0 = 4a + 4 - 2a$

$$-4 = 2a$$
$$-2 = a$$

Finally (2) gives

$$18 = 4a + 2b + c$$
$$18 = 4(-2) + 2b + 10$$
$$18 = -8 + 2b + 10$$
$$16 = 2b$$
$$8 = b$$

$\therefore \boxed{a = -2,\ b = 8,\ c = 10}$

3) $y = ax^4 + bx^3 + cx^2$

3 unknowns : need 3 equations.

$\frac{dy}{dx}$ cannot be used $= 0$ because we don't know if points of inflection are stationary points.

<u>BUT</u> we can do the following

$$\frac{dy}{dx} = 4ax^3 + 3bx^2 + 2cx$$

$$\frac{d^2y}{dx^2} = 12ax^2 + 6bx + 2c = 0$$ at inflection points.

$(0,0)$

$$12a(0^2) + 6(b)(0) + 2(c) = 0$$
$$0 + 0 + 2c = 0$$
$$c = 0 \quad — \quad 1$$

$(\frac{1}{2}, -\frac{1}{16})$

$$12a\left(\tfrac{1}{2}\right)^2 + 6b\left(\tfrac{1}{2}\right) + 2(0) = 0$$
$$3a + 3b = 0$$
$$a + b = 0 \quad — \quad 2$$

<u>ALSO</u>

Both points lie on the curve

Use $(\frac{1}{2}, -\frac{1}{16})$

$$y = ax^4 + bx^3 + cx^2$$

$c = 0$

$$-\frac{1}{16} = a\left(\tfrac{1}{2}\right)^4 + b\left(\tfrac{1}{2}\right)^3 + 0$$
$$-\frac{1}{16} = \frac{a}{16} + \frac{b}{8}$$
$$-1 = a + 2b \quad — \quad 3$$

Solve 2 + 3 simultaneously

② $a = -b$ (*)

Sub (*) into ③

③ $\Rightarrow$ $-1 = -b + 2b$

$-1 = b$

(*) gives

$a = -(-1)$

$a = 1$

$\therefore$ $a = 1$, $b = -1$, $c = 0$

4) $y = ax^3 + bx^2 + c$

We need 3 equations for 3 unknowns

First we can't use $\frac{dy}{dx} = 0$ because we don't know if the inflection point is a stationary point.

BUT we can do the following

$$\frac{dy}{dx} = 3ax^2 + 2bx$$

$$\frac{d^2y}{dx^2} = 6ax + 2b = 0 \quad \text{at inflection points.}$$

At $(4,2)$

$$6a(4) + 2b = 0$$
$$24a + 2b = 0$$
$$-12a = b \quad \text{— } ①$$

Now curve passes through $(4,2)$ and $(0,130)$

$$\therefore \quad y = ax^3 + bx^2 + c$$

$(4,2)$

$$2 = 64a + 16b + c \quad \text{— } ②$$

$(0,130)$ $\quad 130 = 0 + 0 + c$

$$130 = c \quad \text{— } ③$$

sub ③ into ②

$$2 = 64a + 16b + 130$$
$$-128 = 64a + 16b \quad \div 4$$
$$-32 = 16a + 4b \quad \div 4$$
$$-8 = 4a + b$$
$$-4a - 8 = b \quad ** \quad ④$$

Now solve ① and ④ simultaneously.

sub ① into ④

$$-4a - 8 = -12a$$
$$-8 = -8a$$
$$8a = 8$$
$$a = 1$$

from ④

$$b = -4a - 8$$
$$b = -4(1) - 8$$
$$b = -12$$

$\therefore$

$$a = 1 \quad b = -12 \quad c = 130$$

5) $y = x^3 + ax^2 + bx + c$

3 unknowns. Need 3 equations.

First $\frac{dy}{dx} = 3x^2 + 2ax + b = 0$ at SP's

(1,-3) is a SP.

$$3(1^2) + 2a(1) + b = 0$$
$$3 + 2a + b = 0$$
$$b = -3 - 2a \quad — \quad (1)$$

ALSO (1,-3) and (0,-4) lie on the curve

(1,-3) $y = x^3 + ax^2 + bx + c$

$$-3 = 1 + a + b + c$$
$$-4 = a + b + c \quad — \quad (2)$$

(0,-4) $-4 = 0 + 0 + 0 + c$

$$-4 = c \quad — \quad (3)$$

~~Solve~~ Sub (3) into (2)

$$(2) \Rightarrow -4 = a + b - 4$$
$$0 = a + b$$
$$-a = b \quad — \quad (4)$$
or $a = -b$

Now sub (4) into (1)

$$(1) \Rightarrow b = -3 - 2(-b)$$
$$b = -3 + 2b$$
$$3 = b$$

$(4) \Rightarrow a = -3$

$\therefore$ $a = -3$ $b = 3$ $c = -4$