Integration By Substitution 3 : Answers

1) $u = 2\sin x + 3$

$\frac{du}{dx} = 2\cos x$

$\frac{du}{2} = \cos x \, dx$

Limits

$x = 0$

$u = 2\sin 0 + 3$

$u = 3$

$x = \pi/6$

$u = 2\sin \pi/6 + 3$

$u = 1 + 3$

$u = 4$

$$\therefore \int_0^{\pi/6} \frac{\cos x}{(2\sin x + 3)^2} \, dx$$

$$= \int_3^4 \frac{1}{u^2} \frac{du}{2}$$

$$= \frac{1}{2}\int_3^4 u^{-2} \, du$$

$$= \frac{1}{2}\left[\frac{u^{-1}}{-1}\right]_3^4$$

$$= -\frac{1}{2}\left[+\frac{1}{u}\right]_3^4$$

$$= -\frac{1}{2}\left[\frac{1}{4} - \frac{1}{3}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{12}\right]$$

$$= +\frac{1}{24}$$

2) $u = 2\cos x + 1$ $\qquad \frac{du}{dx} = -2\sin x$

$$-\frac{du}{2} = \sin x \, dx$$

Limits

$x = 0$

$u = 2\cos 0 + 1$

$u = 3$

$x = \pi/3$

$u = 2\cos \pi/3 + 1$

$u = 1 + 1 = 2$

$$\therefore \int_0^{\pi/3} \frac{\sin x}{\sqrt{2\cos x + 1}} \, dx$$

$$= \int_3^2 u^{-1/2} \left(-\frac{du}{2}\right)$$

$$= -\frac{1}{2} \int_3^2 u^{-1/2} \, du$$

$$= -\frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right]_3^2$$

$$= -\frac{2}{2} \left[\sqrt{u} \right]_3^2$$

$$= -1 \left[\sqrt{2} - \sqrt{3} \right]$$

$$= \sqrt{3} - \sqrt{2}$$

3) $u = 1 + 3\ln x$ $\qquad \frac{du}{dx} = \frac{3}{x}$

$$\frac{du}{3} = \frac{dx}{x}$$

Limits

$x = 1$

$u = 1 + 3\ln 1$

$u = 1 + 0$

$u = 1$

$x = e$

$u = 1 + 3\ln e$

$u = 1 + 3$

$u = 4$

$$\therefore \int_1^e \frac{1}{x(1 + 3\ln x)} \, dx$$

$$= \int_1^4 \frac{1}{u} \frac{du}{3}$$

$$= \frac{1}{3} \int_1^4 \frac{1}{u} \, du = \frac{1}{3} \left[\ln u \right]_1^4$$

$$= \frac{1}{3} \left[\ln 4 - \ln 1 \right]$$

$$= \frac{1}{3} \ln 4$$

$$= 0.4621 \text{ to 4 d.p.}$$

4) $u = 10\cos x - 1$ $\qquad \frac{du}{dx} = -10\sin x$

$$-\frac{du}{10} = \sin x \, dx$$

Limits

$x = 0$

$u = 10\cos 0 - 1$

$u = 9$

$x = \pi/3$

$u = 10\cos \pi/3 - 1$

$u = 5 - 1$

$u = 4$

$$\therefore \int_0^{\pi/3} \sqrt{10\cos x - 1} \, \sin x \, dx$$

$$= \int_9^4 \sqrt{u} \left(-\frac{du}{10}\right)$$

$$= -\frac{1}{10}\int_9^4 u^{1/2} \, du$$

$$= -\frac{1}{10}\left[\frac{u^{3/2}}{3/2}\right]_9^4$$

$$= -\frac{1}{10} \times \frac{2}{3}\left[\sqrt{u^3}\right]_9^4$$

$$= -\frac{1}{15}\left[8 - 27\right]$$

$$= -\frac{1}{15}\left[-19\right] = +\frac{19}{15}$$

5) $x = 2\sin\theta$ $\qquad \frac{dx}{d\theta} = 2\cos\theta$

$$dx = 2\cos\theta \, d\theta$$

Limits

$x = 0$

$0 = 2\sin\theta$

$0 = \sin\theta$

$\theta = 0$

$x = \sqrt{2}$

$\sqrt{2} = 2\sin\theta$

$\frac{\sqrt{2}}{2} = \sin\theta$

$\theta = \pi/4$

$$\therefore \int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4 - x^2}} \, dx = \int_0^{\pi/4} \frac{4\sin^2\theta}{\sqrt{4 - 4\sin^2\theta}} \times 2\cos\theta \, d\theta$$

$$= 8\int_0^{\pi/4} \frac{\sin^2\theta}{\sqrt{4(1 - \sin^2\theta)}} \times \cos\theta \, d\theta$$

$$= 8\int_0^{\pi/4} \frac{\sin^2\theta}{2\cancel{\cos\theta}} \cancel{\cos\theta} \, d\theta$$

$$= 4\int_0^{\pi/4} \sin^2\theta \, d\theta \quad \text{or} \quad \int_0^{\pi/4} 4\sin^2\theta \, d\theta$$

$\therefore$ $k = 4$ $\qquad a = \pi/4$

Now $4\int_0^{\pi/4} \sin^2\theta \, d\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= 4 \times \frac{1}{2}\int_0^{\pi/4} 1 - \cos 2\theta \, d\theta$

$= 2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$

$= 2\left[\left(\pi/4 - \frac{1}{2}\right) - (0-0)\right]$

$= \frac{\pi}{2} - \frac{1}{4} = 1.548$ to 3 d.p.

6) $x = 3\sin\theta$

$\frac{dx}{d\theta} = 3\cos\theta$

$dx = 3\cos\theta \, d\theta$

Limits

$x = 1.5$

$1.5 = 3\sin\theta$

$0.5 = \sin\theta$

$\theta = \pi/6$

$x = 3$

$3 = 3\sin\theta$

$1 = \sin\theta$

$\theta = \pi/2$

$\therefore \int_{1.5}^{3} \sqrt{9-x^2} \, dx$

$= \int_{\pi/6}^{\pi/2} \sqrt{9 - 9\sin^2\theta} \times 3\cos\theta \, d\theta$

$= 3\int_{\pi/6}^{\pi/2} \sqrt{9(1-\sin^2\theta)} \cos\theta \, d\theta = 3\int_{\pi/6}^{\pi/2} \sqrt{9\cos^2\theta} \cos\theta \, d\theta$

$= 3\int_{\frac{\pi}{6}}^{\pi/2} 3\cos\theta\cos\theta \, d\theta$

$= \int_{\pi/6}^{\pi/2} 9\cos^2\theta \, d\theta$

$\therefore k = 9 \quad a = \pi/6$

$b = \pi/2$

Now $9\int_{\pi/6}^{\pi/2} \cos^2\theta \, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$

$= 9 \times \frac{1}{2}\int_{\pi/6}^{\pi/2} \cos 2\theta + 1 \, d\theta$

$= \frac{9}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_{\pi/6}^{\pi/2} = \frac{9}{2}\left[\left(\frac{\sin \pi}{2} + \frac{\pi}{2}\right) - \left(\frac{\sin \pi/3}{2} + \frac{\pi}{6}\right)\right]$

$= \frac{9}{2}\left[(0 + \pi/2) - \left(\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)\right]$

$= \frac{9}{2}\left[\frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6}\right]$

$= \frac{9}{2}[0.181172] = 0.815$ to 3 d.p.

7) You might be tempted to use $u = 1 - x^2$. However if you try this it is not easy to solve the problem.

* The clue that tells you what to use as a substitution is the $\frac{\pi}{12}$ in the answer.

$\frac{\pi}{12}$ comes from rads and tells use to use a trig substitution.

Limits.

$x = 0$

$0 = \sin\theta$

$\theta = 0$

$x = \frac{1}{2}$

$\frac{1}{2} = \sin\theta$

$\frac{\pi}{6} = \theta$

Let $x = \sin\theta$ $\qquad \frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta \, d\theta$

$$\therefore \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} \, dx$$

$$= \int_0^{\pi/6} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \times \cos\theta \, d\theta$$

$$= \int_0^{\pi/6} \frac{\sin^2\theta}{\cancel{\sqrt{\cos^2\theta}}} \times \cancel{\cos\theta} \, d\theta$$

$$= \int_0^{\pi/6} \sin^2\theta \, d\theta$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$= \frac{1}{2}\int_0^{\pi/6} 1 - \cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{6} - \frac{\sin \pi/3}{2}\right) - (0 - 0)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

QED.