Partial Fractions 4 : Answers

1) $$\frac{x+3}{x^2\cancel{(x-1)}} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)}$$

$$= \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2\cancel{(x-1)}}$$

Let $\underline{x = 0}$

$$0 + 3 = 0 + B(-1) + 0$$
$$-3 = B$$

Let $\underline{x = 1}$

$$1 + 3 = 0 + 0 + C$$
$$4 = C$$

Let $\underline{x = 2}$

$$2 + 3 = A(2)(1) + B(1) + C(4)$$
$$5 = 2A + B + 4C$$
$$5 = 2A - 3 + 16$$
$$-8 = 2A$$
$$-4 = A$$

$$\therefore \frac{x+3}{x^2(x-1)} = -\frac{4}{x} - \frac{3}{x^2} + \frac{4}{(x-1)}$$

2) a) $$\frac{3x}{(1+x)^2\cancel{(2+x)}} = \frac{A}{(1+x)} + \frac{B}{(1+x)^2} + \frac{C}{(2+x)}$$

$$= \frac{A(1+x)(2+x) + B(2+x) + C(1+x)^2}{(1+x)^2\cancel{(2+x)}}$$

Let $\underline{x = -1}$

$$-3 = 0 + B(1) + 0$$
$$-3 = B$$

Let $\underline{x = -2}$

$$-6 = 0 + 0 + C(1)$$
$$-6 = C$$

Let $\underline{x = 0}$

$$0 = A(1)(2) + B(2) + C(1)$$
$$0 = 2A + 2B + C$$
$$0 = 2A - 6 - 6$$
$$6 = A$$

$$\therefore \quad \frac{3x}{(1+x)^2(2+x)} = \frac{6}{(1+x)} - \frac{3}{(1+x)^2} - \frac{6}{(2+x)}$$

b) $$\int_0^1 \frac{3x}{(1+x)^2(2+x)}\,dx = \int_0^1 \frac{6}{(1+x)}\,dx - \int_0^1 3(1+x)^{-2}\,dx - \int_0^1 \frac{6}{(2+x)}\,dx$$

$$= \left[6\ln|1+x| - \frac{3(1+x)^{-1}}{(-1)} - 6\ln|2+x|\right]_0^1$$

$$= \left(6\ln 2 + \frac{3}{2} - 6\ln 3\right) - \left[6\ln 1 + \frac{3}{1} - 6\ln 2\right]$$

$$= 6\ln 2 + \frac{3}{2} - 6\ln 3 - 0 - 3 + 6\ln 2$$

$$= 0.226 \text{ to } 3 \text{ d.p.}$$

3) a) $$f(x) = \frac{8-x-x^2}{x\cancel{(x-2)^2}} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$= \frac{A(x-2)^2 + Bx(x-2) + Cx}{x\cancel{(x-2)^2}}$$

Let $\underline{x=0}$

$$8-0-0 = 4A + 0 + 0$$
$$2 = A$$

Let $\underline{x=2}$

$$8-2-4 = 0 + 0 + 2C$$
$$2 = 2C$$
$$1 = C$$

Let $\underline{x=1}$

$$8-1-1 = A(1) + B(1)(-1) + C(1)$$
$$6 = A - B + C$$
$$6 = 2 - B + 1$$
$$B = -3$$

$$\therefore \quad f(x) = \frac{2}{x} - \frac{3}{(x-2)} + \frac{1}{(x-2)^2}$$

b) $f(x) = 2x^{-1} - 3(x-2)^{-1} + (x-2)^{-2}$

$$f'(x) = -\frac{2}{x^2} + \frac{3}{(x-2)^2} - \frac{2}{(x-2)^3}$$

$$\therefore f'(1) = \frac{-2}{1} + \frac{3}{1} - \frac{2}{(-1)}$$

$$= -2 + 3 + 2$$
$$= 3$$

4) a) $f(x) = \dfrac{x^2+x+13}{(x+2)^2\cancel{(x-3)}} = \dfrac{A}{(x+2)} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x-3)}$

$$= \frac{A(x+2)(x-3) + B(x-3) + C(x+2)^2}{\cancel{(x+2)^2(x-3)}}$$

Let $x = 3$

$$9 + 3 + 13 = 0 + 0 + 25C$$
$$25 = 25C$$
$$1 = C$$

Let $x = -2$

$$4 - 2 + 13 = 0 + B(-5) + 0$$
$$15 = -5B$$
$$-3 = B$$

Let $x = 1$

$$1 + 1 + 13 = A(3)(-2) + B(-2) + C(9)$$
$$15 = -5A - 2B + 9C$$
$$15 = -5A + 6 + 9$$
$$5A = 0$$
$$A = 0$$

$$\therefore f(x) = \frac{-3}{(x+2)^2} + \frac{1}{(x-3)}$$

b) $\displaystyle\int_6^7 f(x)\,dx = \int_6^7 3(x+2)^{-2} + \frac{1}{(x-3)}\,dx$

$$= \left[\frac{-3}{(x+2)} + \ln|x-3|\right]_6^7$$

$$= \left(\frac{-3}{9} + \ln 4\right) - \left(\frac{-3}{8} + \ln 3\right)$$

$$= 0.329 \text{ to } 3 \text{ d.p.}$$

5) $f(x) = \dfrac{11 + x - x^2}{\cancel{(x+1)(x-2)^2}} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$

$$= \frac{A(x-2)^2 + B(x-2)(x+1) + C(x+1)}{\cancel{(x+1)(x-2)^2}}$$

Let $\underline{x = 2}$

$$11 + 2 - 4 = 0 + 0 + 3C$$
$$9 = 3C$$
$$3 = C$$

Let $\underline{x = -1}$

$$11 - 1 - 1 = A(9) + 0 + 0$$
$$9 = 9A$$
$$1 = A$$

Let $\underline{x = 0}$

$$11 + 0 - 0 = A(4) + B(-2)(1) + C(1)$$
$$11 = 4A - 2B + C$$
$$11 = 4 - 2B + 3$$
$$2B = -4$$
$$B = -2$$

$\therefore f(x) = \dfrac{1}{(x+1)} - \dfrac{2}{(x-2)} + \dfrac{3}{(x-2)^2}$

6) a) $f(x) = \dfrac{6+x-9x^2}{x^2\cancel{(x+2)}} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x+2)}$

$$= \frac{Ax(x+2) + B(x+2) + Cx^2}{x^2\cancel{(x+2)}}$$

Let $\underline{x=0}$

$$6+0-0 = 0+2B+0$$
$$6 = 2B$$
$$3 = B$$

Let $\underline{x=-2}$

$$6-2-36 = 0+0+4C$$
$$-32 = 4C$$
$$-8 = C$$

Let $\underline{x=1}$

$$6+1-9 = A(1)(3) + B(3) + C(1)$$
$$-2 = 3A + 3B + C$$
$$-2 = 3A + 9 - 8$$
$$-3 = 3A$$
$$-1 = A$$

$$\therefore f(x) = \frac{-1}{x} + \frac{3}{x^2} - \frac{8}{(x+2)}$$

b) $f(x) = -x^{-1} + 3x^{-2} - 8(x+2)^{-1}$

$$f'(x) = +x^{-2} - 6x^{-3} + 8(x+2)^{-2}$$
$$= \frac{1}{x^2} - \frac{6}{x^3} + \frac{8}{(x+2)^2}$$

c) $\dfrac{1}{x^2} - \dfrac{6}{x^3} + \dfrac{8}{(x+2)^2} = 0$ at s.P.'s.

$\times x^3(x+2)^2$

$$x(x+2)^2 - 6(x+2)^2 + 8x^3 = 0$$
$$x(x^2+4x+4) - 6(x^2+4x+4) + 8x^3 = 0$$
$$x^3+4x^2+4x-6x^2-24x-24+8x^3 = 0$$

(*) $\quad 9x^3 - 2x^2 - 20x - 24 = 0$

for LHS

$$f(+2) = 9(+2)^3 - 2(+2)^2 - 20(+2) - 24$$
$$= +72 - 8 - 40 - 24$$
$$= 0$$

$\therefore$ $(x-2)$ is a factor of LHS.

$\therefore$ $x = 2$ is a soln. of the eqn. (*)

$\therefore$ stationary value at $x = 2$.