Year 13 Assessment Nov 2020 Paper 2

1) $f(x) = \dfrac{x^2+x+13}{(x+2)^2(x-3)}$

a) $f(x) = \dfrac{A}{(x+2)} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x-3)}$

$x^2+x+13 = A(x+2)(x-3) + B(x-3) + C(x+2)^2$

$\underline{x=3}$

$9+3+13 = 0A + 0B + 25C$

$25 = 25C$

$1 = C$

$\underline{x=-2}$

$4-2+13 = 0A + B(-5) + 0C$

$15 = -5B$

$-3 = B$

$\underline{x=0}$

$13 = A(2)(-3) + B(-3) + C(4)$

$13 = -6A - 3B + 4C$

$13 = -6A + 9 + 4$

$6A = 0$

$A = 0$

So $f(x) = \dfrac{-3}{(x+2)^2} + \dfrac{1}{(x-3)}$

b) $\displaystyle\int_0^7 f(x)\,dx$

$= \displaystyle\int_0^7 \frac{-3}{(x+2)^2}\,dx + \int_0^7 \frac{1}{(x-3)}\,dx$

$= \displaystyle -3\int_0^7 (x+2)^{-2}\,dx + \int_0^7 \frac{1}{(x-3)}\,dx$

$= \left[\dfrac{-3(x+2)^{-1}}{-1\times 1} + \ln|x-3|\right]_0^7$

$= \left[\dfrac{+3}{(x+2)} + \ln|x-3|\right]_0^7$

$$= \left(\frac{3}{9} + \ln 4\right) - \left(\frac{3}{2} + \ln 3\right)$$

$$= \frac{1}{3} + \ln 4 - \frac{3}{2} - \ln 3$$

$$= \frac{2}{6} - \frac{9}{6} + \ln 4 - \ln 3$$

$$= -\frac{7}{6} + \ln \frac{4}{3}$$

$$= -0.879 \text{ to 3 d.p.}$$

$$\frac{3}{9} + \ln 4 - \left(\frac{3}{8} + \ln 3\right)$$

$$= \frac{3}{9} - \frac{3}{8} + \ln \frac{4}{3}$$

$$= 0.246$$

2) $x^4 - 2x^2y + y^2 = 4$

$$4x^3 - \left[2x^2\frac{dy}{dx} + 4xy\right] + 2y\frac{dy}{dx} = 0$$

$$4x^3 - 2x^2\frac{dy}{dx} - 4xy + 2y\frac{dy}{dx} = 0$$

$\underline{2x^2y}$

$u = 2x^2 \qquad v = y$

$\frac{du}{dx} = 4x \qquad \frac{dv}{dx} = \frac{dy}{dx}$

$u\frac{dv}{dx} + v\frac{du}{dx}$

$= 2x^2\frac{dy}{dx} + 4xy$

$$2\frac{dy}{dx}(y - x^2) = 4xy - 4x^3$$

$$2\frac{dy}{dx}(y - x^2) = 4x(y - x^2)$$

$$\frac{dy}{dx} = \frac{\cancel{4}^{2}x\cancel{(y-x^2)}}{\cancel{2}\cancel{(y-x^2)}}$$

$$\frac{dy}{dx} = 2x$$

At $(1,3)$ $\quad m = 2(1) = 2$

$\therefore$ Gradient normal at $(1,3) = -\frac{1}{2}$ $\qquad -\frac{1}{2} \times 2 = -1$

$\therefore$ Equation of normal at $(1,3)$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -\frac{1}{2}(x - 1)$$

$\underline{\times 2}$

$$2y - 6 = -1(x - 1)$$

$$2y - 6 = -x + 1$$

$$2y + x = 7$$

3) a) $y = (7-9x^2)^5$

$$\frac{dy}{dx} = 5(7-9x^2)^4 \times (-18x)$$

$$= -90x(7-9x^2)^4$$

b) $y = \tan^{-1} 6x$

$$\frac{dy}{dx} = \frac{1}{1+(6x)^2} \times 6$$

$$= \frac{6}{1+36x^2}$$

c) $y = e^{4x} \tan 2x$

$u = e^{4x}$ $\qquad$ $v = \tan 2x$

$\frac{du}{dx} = 4e^{4x}$ $\qquad$ $\frac{dv}{dx} = 2\sec^2 2x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= 2e^{4x}\sec^2 2x + 4e^{4x}\tan 2x$$

$$= 2e^{4x}(\sec^2 2x + 2\tan 2x)$$

d) $y = \frac{3+\sin x}{2+\cos x}$

$u = 3+\sin x$ $\qquad$ $v = 2+\cos x$

$\frac{du}{dx} = \cos x$ $\qquad$ $\frac{dv}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{1}{v^2}\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]$$

$$= \frac{1}{(2+\cos x)^2}\left[(2+\cos x)\cos x - (3+\sin x)(-\sin x)\right]$$

$$= \frac{1}{(2+\cos x)^2}\left[2\cos x + \cancel{\cos^2 x} + 3\sin x + \cancel{\sin^2 x}\right]$$

$$1$$

$$= \frac{1}{(2+\cos x)^2}\left[2\cos x + 3\sin x + 1\right]$$

$$= \frac{(2\cos x + 3\sin x + 1)}{(2+\cos x)^2}$$

4) a) (i) $\int \cos\left(3x + \frac{\pi}{2}\right) dx$

$$= \frac{1}{3} \sin\left(3x + \pi/2\right) + C$$

(ii) $\int e^{3-4x} \, dx$

$$= \frac{e^{3-4x}}{-4} + C$$

$$= -\frac{e^{3-4x}}{4} + C$$

(iii) $\int \frac{7}{8x+5} \, dx$

$$= 7 \int \frac{1}{8x+5} \, dx$$

$$= \frac{7}{8} \int \frac{8}{8x+5} \, dx$$

$$= \frac{7}{8} \ln|8x+5| + C$$

b) $\int_1^2 \frac{9}{(2x-1)^4} \, dx$

$$= 9 \int_1^2 (2x-1)^{-4} \, dx$$

$$= 9 \left[\frac{(2x-1)^{-3}}{-3 \times 2} \right]_1^2$$

$$= 9 \left[-\frac{1}{6(2x-1)^3} \right]_1^2$$

$$= -\frac{3}{2} \left[\frac{1}{(2x-1)^3} \right]_1^2$$

$$= -\frac{3}{2} \left[\left(\frac{1}{27} \right) - \left(\frac{1}{1} \right) \right]$$

$$= -\frac{3}{2} \left[-\frac{26}{27} \right]$$

$$= \frac{13}{9}$$

5) a) $\frac{dN}{dt} \propto N$

$$\frac{dN}{dt} = kN$$

b) $\frac{dN}{dt} = kN$

$$\int \frac{dN}{N} = k\int dt$$

$$\ln N = kt + c$$

Now at $\left.\begin{array}{l} t=0 \\ N=A \end{array}\right\}$

$$\ln A = 0 + c$$

$$\ln A = c$$

$\therefore$ particular solution

$$\ln N = kt + \ln A$$

$$\ln N - \ln A = kt$$

$$\ln N/A = kt$$

$$e^{kt} = \frac{N}{A}$$

$$Ae^{kt} = N$$

$$\underline{N = Ae^{kt}}$$

c) $\left.\begin{array}{l} t=2 \\ N=100 \end{array}\right\}$ $100 = Ae^{2k}$ — 1

$\left.\begin{array}{l} t=12 \\ N=160 \end{array}\right\}$ $160 = Ae^{12k}$ — 2

Solve 1 + 2

1 $\frac{100}{e^{2k}} = A$ (*)

Sub into 2

2 $\Rightarrow$ $160 = \frac{100}{e^{2k}} e^{12k}$ (2nd law of indices)

$$160 = 100e^{10k}$$

$$\frac{8}{5} = e^{10k}$$

$$\ln\frac{8}{5} = \ln e^{10k}$$

$$\ln\frac{8}{5} = 10k \cancel{\ln e}$$

$$\frac{1}{10}\ln\frac{8}{5} = k$$

$k = 0.047$ to 3 dp

$$A = \frac{100}{e^{2(0.047)}}$$

$A = 91.028$

(ii) $t = 20$
$k = 0.047$
$A = 91.028$

$$N = Ae^{kt}$$

$$N = 91.028e^{0.047(20)}$$

$$N = 233$$

6) a) $\int x e^{-2x}\, dx$

$u = x$ $\quad \frac{dv}{dx} = e^{-2x}$

$\frac{du}{dx} = 1$ $\quad v = -\frac{e^{-2x}}{2}$

$$\int u\frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx$$

$$= -\frac{xe^{-2x}}{2} - \int -\frac{e^{-2x}}{2}(1)\, dx$$

$$= -\frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\, dx$$

$$= -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + c$$

$$= -\frac{e^{-2x}}{2}\left(x + \frac{1}{2}\right) + c$$

$$= -\frac{e^{-2x}}{2}\left(\frac{2x+1}{2}\right) + c$$

$$= \frac{-e^{-2x}}{4}(2x+1) + c$$

6) b) $u = 1 + 3\ln x$

Limits

$x = 1$: $u = 1 + 3\ln 1$, $u = 1$

$x = e$: $u = 1 + 3\ln e$, $u = 4$

Let $I = \int_1^e \frac{1}{x(1+3\ln x)}\,dx$

$$\frac{du}{dx} = \frac{3}{x}$$

$$\frac{du}{3} = \frac{dx}{x}$$

$$\therefore \quad I = \frac{1}{3}\int_1^4 \frac{1}{u}\,du$$

$$= \frac{1}{3}\left[\ln u\right]_1^4$$

$$= \frac{1}{3}\left[\ln 4 - \ln 1\right]$$

$$= \frac{1}{3}\ln 4$$

$$= 0.4621 \text{ to 4 d.p.}$$

7) a) $-\frac{dv}{dt} \propto v^3$

$$\frac{dv}{dt} = -kv^3$$

b) $\int \frac{dv}{v^3} = -k\int dt$

$$\int v^{-3}\,dv = -k\int dt$$

$$\frac{v^{-2}}{-2} = -kt + c$$

$$-\frac{1}{2v^2} = -kt + c$$

$t=0 \quad v=60$

$$-\frac{1}{7200} = 0 + c$$

$$-\frac{1}{7200} = c$$

$\therefore$ Part soln $\quad -\frac{1}{2v^2} = -kt - \frac{1}{7200}$

$$\frac{1}{7200} + kt = \frac{1}{2v^2}$$

$\underline{\times 2}$ $\quad \frac{1}{3600} + 2kt = \frac{1}{v^2}$

$$\frac{1}{3600} + \frac{7200kt}{3600} = \frac{1}{v^2}$$

$$\frac{1 + 7200kt}{3600} = \frac{1}{v^2}$$

$$\frac{3600}{1 + 7200kt} = v^2$$

$$\frac{3600}{at + 1} = v^2$$

where $\quad a = 7200k$

c) $t = 2 \quad V = 50$

$$V^2 = \frac{3600}{at+1}$$

$$2500 = \frac{3600}{2a+1}$$

$$2a+1 = \frac{3600}{2500}$$

$$2a+1 = \frac{36}{25}$$

$$2a = \frac{36}{25} - \frac{25}{25}$$

$$2a = \frac{11}{25}$$

$$a = \frac{11}{50}$$

$V = 27$

$a = 11/50$

$$27^2 = \frac{3600}{\frac{11}{50}t+1}$$

$$27^2\left(\frac{11}{50}t+1\right) = 3600$$

$$\cancel{5.94t + 27 = 3600}$$

$$\cancel{5.94t = 3600 - 27}$$

$$\cancel{t = \frac{3600-27}{5.94}}$$

$$\cancel{t = 601.6 \text{ hours}}$$

$$\frac{11}{50}t + 1 = 4.938$$

$$\frac{11}{50}t = 3.938$$

$$t = \frac{3.938 \times 50}{11}$$

$t = 17.9$ hours.